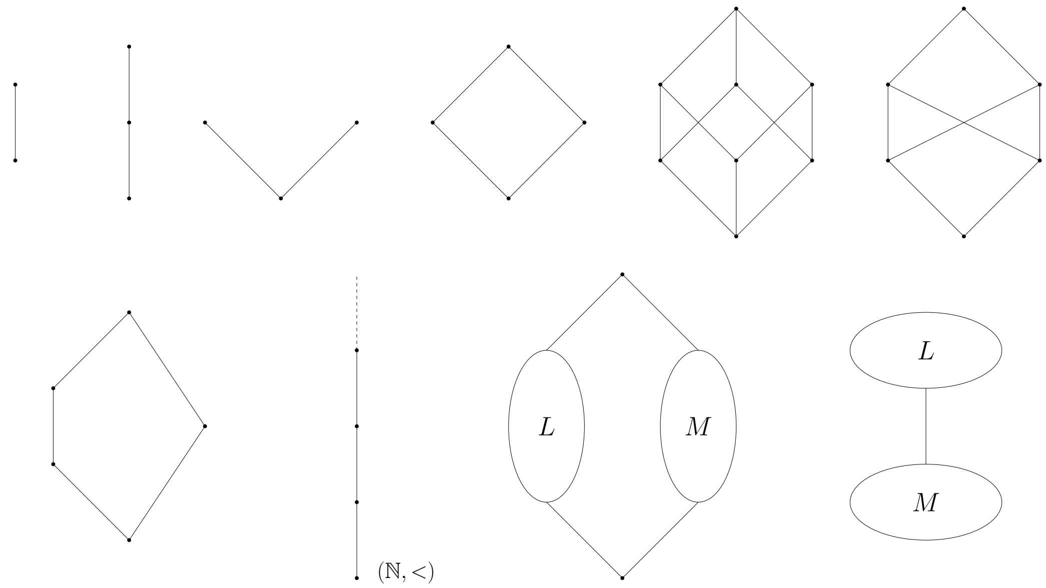
Exercises for Universal Algebra 1

Exercise 1. Which of the following Hasse Diagrams represent (semi)lattices, given that L and M are lattices?



Exercise 2. Find the correspondence between lattices and lattice ordered sets:

(i) For a given semilattice $(S, \wedge)$ find a natural definition of a semilattice order on S, and vice-versa.

(ii) Conclude that there is a 1-to-1 correspondence between semilattices and semilattice ordered sets.

(iii) Prove that there is a 1-to-1 correspondence between lattices and lattice ordered sets.

Exercise 3. Show that every homomorphism of lattices is order preserving. What about the converse?

Exercise 4. Choose your favorite (finite) group and draw the poset of its normal subgroups, ordered by inclusion of their underlying sets.

Exercise 5. Given a group G, prove that the poset of its normal subgroups is a lattice (ordered set). What are the operations meet and join?

Exercise 6. Let $(P, \leq)$ be a poset. Show that there is a linear order $\leq'$ on P such that $p \leq q \implies p \leq' q$ for all $p, q \in P$.

Exercise 7 (*)**.** Let $(P, \leq)$ be a poset, $(\mathcal{U}_P, \subseteq)$ the poset of its upsets and $(\mathcal{D}_P, \subseteq)$ the poset of its downsets.

(i) Find an injective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(ii) Show that there is no surjective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(iii) Conclude that for any set X, there is no surjective map $X \to 2^X$

Exercise 8. Prove or disprove that for every poset $(P, \leq)$ we have

$$(\mathcal{D}_P, \subseteq) \cong (\mathcal{U}_P, \supseteq)$$

Definition. A lattice L is called *distributive*, if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{1.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{1.2}$$

Exercise 9. Show that every lattice with less than four elements is distributive. Find examples of distributive lattices with a large number of elements.

Exercise 10. Show that in the definition of distributive lattices (3.1) and (3.2) are equivalent.

Definition. A lattice L is called *modular*, if for all $x, y, z \in L$

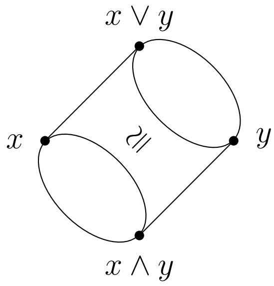
$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z$$

Exercise 11. Show that every distributive lattice is modular and disprove the converse, i.e. find a modular lattice that is not distributive.

Exercise 12. Show that the following two statements hold for all lattices L and all $x, y, z \in L$

$$
\begin{aligned}
x \vee (y \wedge z) &\leq (x \vee y) \wedge (x \vee z) \\
x \leq z \implies x \vee (y \wedge z) &\leq (x \vee y) \wedge z
\end{aligned}
$$

Exercise 13 (Diamond isomorphism theorem)**.** Let L be a modular lattice and $x, y \in L$. Show that the intervals $I[x \wedge y, x]$ and $I[y, x \vee y]$ are isomorphic lattices.

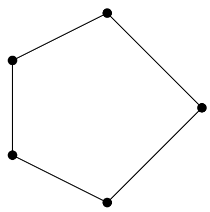


Exercise 14. A term $m(x, y, z)$ of an algebra A is called *majority* if it satisfies the identities

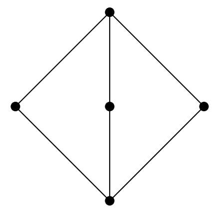
$$x \approx m(x, x, y) \approx m(x, y, x) \approx m(y, x, x)$$

Show that every lattice has a majority term.

Theorem (Dedekind)**.** Prove that a lattice is modular if and only if it does not contain the following lattice as a sublattice.



Theorem (Birkhoff)**.** A modular lattice is distributive if and only if it does not contain the following lattice as a sublattice.



Exercise 15. Show that every complete lattice is bounded.

Exercise 16. Find examples of lattices L that contain a sublattice S such that

(i) L is complete but S is not complete

(ii) L is not complete but S is complete

(iii) both L and S are complete lattices but S is not a complete sublattice

Exercise 17. Let L be a complete lattice and $a, b \in L$ two compact elements.

(i) Is $a \vee b$ compact?

(ii) Is $a \wedge b$ compact?

Exercise 18. Let C be a closure operator on a set X. Prove that L_C is closed under finite unions if and only if for all subsets $U, V \in 2^X$

$$C(U \cup V) = C(U) \cup C(V)$$

Exercise 19. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

(i) Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \Big\{\{1, 2\}, \{2, 3\}\Big\}$ and $A = \Big\{\{1, 2\}, \{2\}\Big\}$. Compare the results.

(ii) Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.

(iii) Prove that for every $A \subseteq 2^X$ we have

$$A^{\to\leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}$$

Exercise 20. Let C be a closure operator on a set X. Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$C(U) = U^{\to\leftarrow}$$

for all subsets $U \subseteq X$.

Exercise 21. Let $\mathbb{A} = (A, *)$ be a binary algebra and θ an equivalence relation on A. Show that θ is a congruence relation if and only if for all $a, b, c \in A$ we have

$$(a, b) \in \theta \implies \begin{cases} (a * c, b * c) \in \theta & \text{and} \\ (c * a, c * b) \in \theta \end{cases}$$

Exercise 22. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

*	0	1	2	3
0	0	2	1	1
1	2	1	0	2
2	1	0	2	0
3	1	2	0	3

Draw the lattice of subalgebras and the lattice of congruences of $\mathbb{A}$.

Exercise 23. Consider the algebra $(\mathbb{Z}, +, \cdot) \times (\mathbb{Z}, \cdot, +)$. What is the subalgebra generated by the pairs $(0, 1)$ and $(1, 0)$?

Exercise 24. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?

- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $f(\theta) \in \mathrm{Con}(\mathbb{B})$ and $f^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?

- Given a subset $X \subseteq A$ is $f(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(f(X))$?

Exercise 25. Given a binary algebra $\mathbb{A} = (A, *)$ define its *nucleus* as

$$B := \{a \in A \mid \forall x, y \in A, (x * a) * y = x * (a * y)\}$$

Show that B is a subalgebra of $\mathbb{A}$ and find and an example of an algebra $\mathbb{A}$ whose nucleus is empty.

Exercise 24. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?

- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $f(\theta) \in \mathrm{Con}(\mathbb{B})$ and $f^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?

- Given a subset $X \subseteq A$ is $f(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(f(X))$?

Exercise 26. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras of the same type and let $f : A \to B$ be a map. Show that f is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if its graph is a subalgebra of $\mathbb{A} \times \mathbb{B}$.

$$\{(a, f(a)) \mid a \in A\} \leq \mathbb{A} \times \mathbb{B}$$

Exercise 27. Let $f, g : \mathbb{A} \to \mathbb{B}$ be two homomorphisms and let $X \subseteq A$ with $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$. Show that

$$f|_X = g|_X \implies f = g.$$

Remark. The converse is also true: Given $\mathbb{A}$ and $X \subseteq A$, then $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$ if and only if $f|_X = g|_X$ implies $f = g$ for all algebras $\mathbb{B}$ and all homomorphisms $f, g : \mathbb{A} \to \mathbb{B}$.

Exercise 28. Let $X \subseteq A$ be a subset that generates the algebra $\mathbb{A}$ such that no proper subset of X generates $\mathbb{A}$. Is it true that every map $f : X \to B$ to any algebra $\mathbb{B}$ can be extended to an homomorphism $\mathbb{A} \to \mathbb{B}$?

Exercise 29. Find all homomorphisms $(\mathbb{N}, +)^2 \to (\mathbb{Z}_2, +)$.

Exercise 30. Show that a map $f : A \to B$ is injective if and only if its kernel is the equality relation.

Exercise 31 (Second isomorphism theorem)**.** Let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

(i) if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism $h : \mathbb{B} \to \mathbb{C}$ such that $g = h \circ f$.

$$\begin{tikzcd} \mathbb{A} \arrow[r, two heads, "f"] \arrow[dr, "g"'] & \mathbb{B} \arrow[d, "\exists h"] \\ & \mathbb{C} \end{tikzcd}$$

(ii) there is an embedding $\mathbb{A}/f^{-1}(\phi) \to \mathbb{B}/\phi$.

(iii) there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta = (\mathbb{A}/\alpha)\big/(\beta/\alpha).$$

Exercise 32. Find classes of algebras witnessing that

$$\mathrm{PS} \lneq \mathrm{SP} \qquad \mathrm{PH} \lneq \mathrm{HP} \qquad \mathrm{SH} \lneq \mathrm{HS}$$