CSP lecture 25/26 – Problem Set 6

A *diagram* D is a directed (multi-)graph G together with an assignment D that assigns to every vertex $v \in G$ a set D_v and to every edge $e\colon v \to w$ a map $D_e\colon D_v \to D_w$. A *solution* to D is a tuple $(d_v)_{v\in G}$ where $d_v \in D_v$ and $D_e(d_v) = d_w$ for every edge $e\colon v \to w$. The problem of deciding whether a given diagram has a solution is called *label cover.*

Problem 0. For any relational structure $\mathbb{A}$ and any instance I of CSP($\mathbb{A}$), find a diagram D such that there is a 1-to-1 correspondence between solutions of D and homomorphisms $I \to \mathbb{A}$.

A diagram D is called *arc-consistent* if all maps D_e are surjective.

Problem 1. Find a polynomial time algorithm that, given a diagram D, finds a maximal arc-consistent subinstance[1] $\mathrm{arc}(D) \subseteq D$. This algorithm is called *arc-consistency.*

$$\begin{array}{ccc} D_v & \xrightarrow{\;D_e\;} & D_w \\ \cup\!| & & \cup\!| \\ \mathrm{arc}(D)_v & \underset{D_e}{\twoheadrightarrow} & \mathrm{arc}(D)_w \end{array}$$

Problem 1.1. Find an unsolvable arc-consistent diagram.

Problem 2. Let $\mathbb{H} = (\{0,1\}; H, \{0\}, \{1\})$, where $H = \{0,1\}^3 \setminus \{(110)\}$ is the HORN-relation. Let I be an instance of CSP($\mathbb{H}$) and let D be the corresponding diagram (see Problem 0). Show that I is a NO-instance if and only if one of the sets in $\mathrm{arc}(D)$ is empty. We say that CSP($\mathbb{H}$) is solved by arc-consistency.

Problem 3. Solve Problem 2 using polymorphisms.

Recall that a *semilattice operation* on a set A is a binary operation $\wedge$ that is associative, commutative, and idempotent: that is, for all $a, b, c \in A$, the following equalities hold:

$$\begin{aligned} (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\ a \wedge b &= b \wedge a \\ a \wedge a &= a \end{aligned}$$

Problem 3.1. Let A be a relational structure such that Pol(A) contains a semilattice operation. Show that arc-consistency solves CSP(A).

We call an operation $t\colon A^n \to A$ *totally symmetric* if $t(a_1, \ldots, a_n) = t(b_1, \ldots, b_n)$ whenever $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\}$, i.e. the value of the operation depends only on the set of its arguments. Observe that every clone that contains a semilattice operation also contains totally symmetric operations of all arities.

Problem 3.2. Let $\mathbb{A}$ be a finite relational structure such that Pol(A) contains a totally symmetric operation of large enough arity. Show that arc-consistency solves CSP($\mathbb{A}$). What is "large enough"?

Problem 4. Let $\mathbb{A}$ be a finite relational structure such that CSP($\mathbb{A}$) is solved by arc-consistency. Show that Pol($\mathbb{A}$) contains totally symmetric operations of all arities. Hint: Consider the instance $\mathbb{A}^n/\sim$, where $(a_1, \ldots, a_n) \sim (b_1, \ldots, b_n)$ if and only if $\{a_1, \ldots a_n\} = \{b_1, \ldots, b_n\}$.

[1] Same underlying graph, $\mathrm{arc}(D)_v \subseteq D_v$ for all vertices v, with the same maps but restricted