Universal Algebra 2 - Exercises 1

Exercise 1.1. Show that an abelian algebra $\mathbb{A}$ satisfies the term condition

$$t(x,\bar{u}) \approx t(x,\bar{v}) \implies t(y,\bar{u}) \approx t(y,\bar{v}) \tag{1.1}$$

not only for term operations t, but also for all polynomials $p \in \mathrm{Clo}(\mathbb{A} +$ constants). Also, show that it is not enough to satisfy (1.1) only in the case where t is a basic operation of $\mathbb{A}$.

Exercise 1.2. Show that a finite monoid $(M, \cdot, 1)$ is abelian if and only if the multiplication $\cdot$ is a commutative group operation. What if M is infinite?

Exercise 1.3. Let A be a 4-element set, fix $0 \in A$ and let $(A, +_1) \cong \mathbb{Z}_4$ and $(A, +_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be the two abelian group operations on A with neutral element 0. Show that $(A, +_1, +_2)$ is not an abelian algebra.

Exercise 1.4. Let $(R, +, 0, -, \cdot)$ be a commutative ring. Recall that congruences α are one-to-one with ideals I, using $I_\alpha = [0]_\alpha$. Show that α centralizes β if and only if $I_\alpha \cdot I_\beta = 0$. More generally, show that $I_\alpha \cdot I_\beta = I_{[\alpha,\beta]}$.

Exercise 1.5. Show the following properties of the centralizer relation C:

- $C(\alpha, \beta; \alpha)$ and $C(\alpha, \beta; \beta)$
- Let Γ be a set of congruences. If $C(\alpha, \beta; \gamma)$ for all $\gamma \in \Gamma$, then $C(\alpha, \beta; \bigwedge \Gamma)$.