Universal Algebra Exercises - Sheet 1

Exercise 1.1. Recall the notion of normal subgroups. Choose your favorite (finite) group and draw the poset of its normal subgroups, ordered by inclusion of their underlying sets.

Exercise 1.2. Let G be a group and let $N, H \trianglelefteq G$ be two normal subgroups. Show that

(i). $N \cap H$ is a normal subgroup.

(ii). $NH := \{n_1 h_1 n_2 h_2 \cdots n_k h_k \mid n_\bullet \in N, h_\bullet \in H\}$ is a normal subgroup.

Exercise 1.3. Choose your favorite (finite) commutative ring and draw the poset of its ideals, ordered by inclusion of their underlying sets.

Exercise 1.4. Recall the notion commutative rings and their ideals. Let R be a commutative ring and let $I, J \trianglelefteq R$ be two ideals. Show that

(i). $I \cap J$ is an ideal.

(ii). $IJ := \{i_1 j_1 + \cdots + i_n j_n \mid i_\bullet \in I, j_\bullet \in J\}$ is an ideal.

Exercise 1.5. Explicitly describe the notions of Exercise 1.4 for the ring $R = (\mathbb{Z}, 0, 1, +, -, \cdot)$.

Definition. A lattice is a triple $(L, \wedge, \vee)$, where L is a set, $\wedge$ ("meet") and $\vee$ ("join") are two binary associative symmetric idempotent operations satisfying the following absorption laws for all elements $x, y \in L$.

$$x \wedge (x \vee y) = x \qquad x \vee (x \wedge y) = x$$

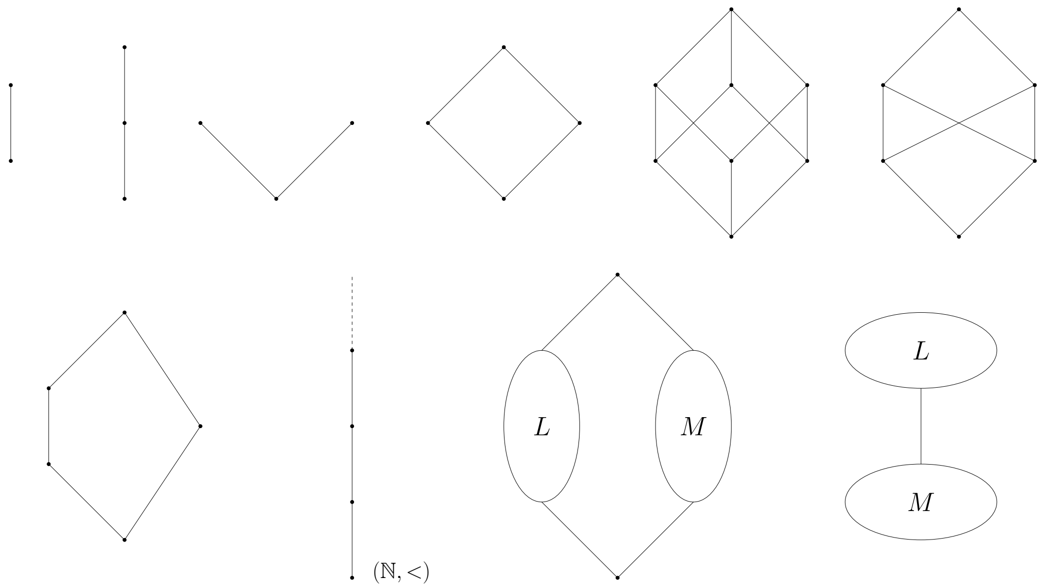
Exercise 1.6. Let X be a set and 2^X its powerset. Show that $(2^X, \cap, \cup)$ forms a lattice.

Exercise 1.7. Let G be a group and $\mathcal{N}_G$ the set of its normal subgroups. Define lattice operations on $\mathcal{N}_G$.

Exercise 1.8. Let R be a commutative ring and $\mathcal{I}_R$ be the set of its ideals. Define lattice operations on $\mathcal{I}_R$.

Universal Algebra Exercises - Sheet 2

Exercise 2.1. Which of the following Hasse Diagrams represent lattices, given that L and M are lattices?



Exercise 2.2. Let $(S, \leq)$ be a poset, such that every pair of elements has a greatest lower bound. Define a semilattice operation on S.

Exercise 2.3. A lattice ordered set is a poset $(L, \leq)$, where every pair of elements has both a greatest lower bound and a smallest upper bound. Show that there is a one to one correspondence between lattice ordered sets and lattices.

Exercise 2.4. Show that every homomorphism of lattices is order preserving. What about the converse?

Exercise 2.5. Let $(P, \leq)$ be a poset, $(\mathcal{U}_P, \subseteq)$ the poset of its upsets and $(\mathcal{D}_P, \subseteq)$ the poset of its downsets.

(i). Show that there is an injective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(ii). Show that there is no surjective order preserving map $(P, \leq) \to (\mathcal{D}_P, \subseteq)$

(iii). Conclude that for any set X, there is no surjective map $X \to 2^X$

Exercise 2.6. Let $(P, \leq)$ be a poset. Show that there is a linear order $\leq'$ on P such that $p \leq q \implies p \leq' q$ for all $p, q \in P$.

Exercise 2.7. Prove or disprove that for every poset $(P, \leq)$ we have

$$(\mathcal{D}_P, \subseteq) \cong (\mathcal{U}_P, \supseteq)$$

Definition. Given a signature τ and two τ-algebras $\mathbb{A} = (A, f^{\mathbb{A}} \mid f \in \tau)$ and $\mathbb{B} = (B, f^{\mathbb{B}} \mid f \in \tau)$, a *homomorphism* $\mathbb{A} \to \mathbb{B}$ is a map $\phi : A \to B$ such that for all symbols $f \in \tau$ and all $a_i \in A$ we have

$$\phi\Big(f^{\mathbb{A}}(a_1, \ldots, a_n)\Big) = f^{\mathbb{B}}\Big(\phi(a_1), \ldots, \phi(a_n)\Big)$$

A homomorphism $f : \mathbb{A} \to \mathbb{B}$ is called *isomorphism* if there is a homomorphism $g : \mathbb{B} \to \mathbb{A}$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Exercise 2.8. Show that

(i). the identity map is a homomorphism

(ii). the composition of two homomorphisms is a homomorphism

(iii). a homomorphism is an isomorphism if and only if it is bijective

Exercise 2.9. Which of the following algebras are isomorphic?

(i). $(\mathbb{Q}, +, 0)$ and $(\mathbb{R}, +, 0)$

(ii). $(\mathbb{C}, +)$ and $(\mathbb{R}^2, +)$

(iii). $(\mathbb{C}, \cdot)$ and $(\mathbb{R}^2, \cdot)$

(iv). $(\mathbb{N}, \cdot)$, $(2\mathbb{N}, \cdot)$ and $(3\mathbb{N}, \cdot)$

(v). $(a\mathbb{N}, \cdot)$ and $(b\mathbb{N}, \cdot)$

Universal Algebra Exercises - Sheet 3

Definition. A lattice L is called *distributive*, if for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{3.1}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{3.2}$$

Exercise 3.1. Show that every lattice with less than four elements is distributive. Find examples of distributive lattices with a large number of elements.

Exercise 3.2. Show that in the definition of distributive lattices (3.1) and (3.2) are equivalent.

Definition. A lattice L is called *modular*, if for all $x, y, z \in L$

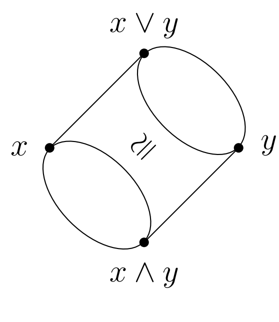
$$x \leq z \implies x \vee (y \wedge z) = (x \vee y) \wedge z$$

Exercise 3.3. Show that every distributive lattice is modular and disprove the converse, i.e. find a modular lattice that is not distributive.

Exercise 3.4. Show that the following two statements hold for all lattices L and all $x, y, z \in L$

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$
$$x \leq z \implies x \vee (y \wedge z) \leq (x \vee y) \wedge z$$

Exercise 3.5 (Diamond isomorphism theorem)**.** Let L be a modular lattice and $x, y \in L$. Show that the intervals $I[x \wedge y, x]$ and $I[y, x \vee y]$ are isomorphic lattices.



Exercise 3.6. A term $m(x, y, z)$ of an algebra A is called *majority* if it satisfies the identities

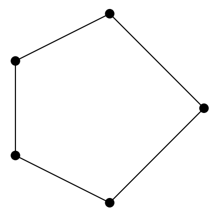
$$x \approx m(x, x, y) \approx m(x, y, x) \approx m(y, x, x)$$

Show that every lattice has a majority term.

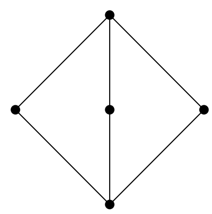
Exercise 3.7. Let G be a group and $\mathcal{S}_G$ and $\mathcal{N}_G$ the lattices of its subgroups and normal subgroups respectively. Decide whether $\mathcal{S}_G$ and $\mathcal{N}_G$ are modular, distributive or neither.

Exercise 3.8. Let R be a ring and $\mathcal{I}_R$ the lattices of its Ideals. Decide whether $\mathcal{I}_R$ is modular, distributive or neither.

Exercise 3.9 (Dedekind)**.** Prove that a lattice is modular if and only if it does not contain the following lattice as a sublattice.



Theorem (Birkhoff)**.** A modular lattice is distributive if and only if it does not contain the following lattice as a sublattice.



Universal Algebra Exercises - Sheet 4

Exercise 4.1. Show that every complete lattice is bounded.

Exercise 4.2. Find examples of lattices L that contain a sublattice S such that

(i). L is complete but S is not complete

(ii). L is not complete but S is complete

(iii). both L and S are complete lattices but S is not a complete sublattice

Exercise 4.3. Let L be a complete lattice and $a, b \in L$ two compact elements.

(i). Is $a \vee b$ compact?

(ii). Is $a \wedge b$ compact?

Exercise 4.4. Let C be a closure operator on a set X. Prove that L_C is closed under finite unions if and only if for all subsets $U, V \in 2^X$

$$C(U \cup V) = C(U) \cup C(V)$$

Exercise 4.5. Let X be a set and let ϕ be the binary relation on 2^X defined by

$$(U, V) \in \phi \iff U \cap V \neq \emptyset$$

Consider the Galois correspondence on the sets 2^{2^X} and 2^{2^X} induced by this relation

(i). Let $X = \{1, 2, 3, 4\}$. Compute $A^{\leftarrow\rightarrow}$ and $A^{\rightarrow\leftarrow}$ for both $A = \Big\{\{1, 2\}, \{2, 3\}\Big\}$ and $A = \Big\{\{1, 2\}, \{2\}\Big\}$. Compare the results.

(ii). Prove that if a Galois correspondence is defined by a symmetric relation on a set, then the closure operators induced by it coincide.

(iii). Prove that for every $A \subseteq 2^X$ we have

$$A^{\rightarrow\leftarrow} = \{U \in 2^X \mid \exists V \in A, V \subseteq U\}$$

Exercise 4.6. Let C be a closure operator on a set X. Find a relation $\phi \subseteq X \times 2^X$ whose induced Galois correspondence gives

$$C(U) = U^{\rightarrow\leftarrow}$$

for all subsets $U \subseteq X$.

Universal Algebra Exercises - Sheet 5

Exercise 5.1. Let $\mathbb{A} = (A, *)$ be a binary algebra and θ an equivalence relation on A. Show that θ is a congruence relation if and only if for all $a, b, c \in A$ we have

$$(a,b) \in \theta \implies \begin{cases} (a * c, b * c) \in \theta & \text{and} \\ (c * a, c * b) \in \theta \end{cases}$$

Exercise 5.2. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

$*$	0	1	2	3
0	o	2	1	1
1	2	1	0	2
2	1	0	2	0
3	1	2	0	3

Draw the lattice of subalgebras and the lattice of congruences of $\mathbb{A}$.

Exercise 5.3. Consider the algebra $(\mathbb{Z}, +, \cdot) \times (\mathbb{Z}, \cdot, +)$. What is the subalgebra generated by the pairs $(0, 1)$ and $(1, 0)$?

Exercise 5.4. Let $\mathbb{B} = (\{0, 1\}, \wedge, \vee, \neg, 0, 1)$ be the two element Boolean algebra. Show that for every set X

$$(2^X, \cap, \cup, X \setminus (-), \emptyset, X) \cong \mathbb{B}^X$$

Exercise 5.5. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras in the same signature and let $f : \mathbb{A} \to \mathbb{B}$ be a homomorphism.

- Given two subalgebras $U \leq \mathbb{A}$ and $V \leq B$, are $f(U) \subseteq \mathbb{B}$ and $f^{-1}(V) \subseteq \mathbb{A}$ subalgebras?

- Given two congruences $\theta \in \mathrm{Con}(\mathbb{A})$ and $\psi \in \mathrm{Con}(\mathbb{B})$, is $h(\theta) \in \mathrm{Con}(\mathbb{B})$ and $h^{-1}(\psi) \in \mathrm{Con}(\mathbb{A})$?

- Given a subset $X \subseteq A$ is $h(\mathrm{Sg}_{\mathbb{A}}(X)) = \mathrm{Sg}_{\mathbb{B}}(h(X))$?

Exercise 5.6. Given a binary algebra $\mathbb{A} = (A, *)$ define its *nucleus* as

$$B := \{a \in A \mid \forall x, y \in A, (x * a) * y = x * (a * y)\}$$

Show that B is a subalgebra of $\mathbb{A}$ and find and an example of an algebra $\mathbb{A}$ whose nucleus is empty.

Universal Algebra Exercises - Sheet 6

Exercise 6.1. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras of the same type and let $f : A \to B$ be a map. Show that f is a homomorphism from $\mathbb{A}$ to $\mathbb{B}$ if and only if its graph is a subalgebra of $\mathbb{A} \times \mathbb{B}$.

$$\{(a, f(a)) \mid a \in A\} \leq \mathbb{A} \times \mathbb{B}$$

Exercise 6.2. Let $f, g : \mathbb{A} \to \mathbb{B}$ be two homomorphisms and let $X \subseteq A$ with $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$. Show that

$$f|_X = g|_X \implies f = g.$$

Remark. The converse is also true: Given $\mathbb{A}$ and $X \subseteq A$, then $\mathbb{A} = \mathrm{Sg}_{\mathbb{A}}(X)$ if and only if $f|_X = g|_X$ implies $f = g$ for all algebras $\mathbb{B}$ and all homomorphisms $f, g : \mathbb{A} \to \mathbb{B}$.

Exercise 6.3. Let $X \subseteq A$ be a subset that generates the algebra $\mathbb{A}$ such that no proper subset of X generates $\mathbb{A}$. Is it true that every map $f : X \to B$ to any algebra $\mathbb{B}$ can be extended to an homomorphism $\mathbb{A} \to \mathbb{B}$?

Exercise 6.4. Find all homomorphisms $(\mathbb{N}, +)^2 \to (\mathbb{Z}_2, +)$.

Exercise 6.5. Show that a map $f : A \to B$ is injective if and only if its kernel is the equality relation.

Exercise 6.6 (Second isomorphism theorem)**.** Let $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{A} \to \mathbb{C}$ be two homomorphisms and let $\alpha \leq \beta$ be two congruences on $\mathbb{A}$ and let ϕ be a congruence on $\mathbb{B}$. Prove that

(i). if f is surjective and $\ker(f) \subseteq \ker(g)$, then there exists a homomorphism $h : \mathbb{B} \to \mathbb{C}$ such that $g = h \circ f$.

$$\begin{tikzcd} \mathbb{A} \arrow[r, two heads, "f"] \arrow[dr, "g"'] & \mathbb{B} \arrow[d, "\exists h"] \\ & \mathbb{C} \end{tikzcd}$$

(ii). there is an embedding $\mathbb{A}/f^{-1}(\phi) \to \mathbb{B}/\phi$.

(iii). there is a congruence β/α on $\mathbb{A}/\alpha$ such that

$$\mathbb{A}/\beta = (\mathbb{A}/\alpha)\big/(\beta/\alpha).$$

Exercise 6.7. Find classes of algebras witnessing that

$$\mathrm{PS} \lneq \mathrm{SP} \qquad \mathrm{PH} \lneq \mathrm{HP} \qquad \mathrm{SH} \lneq \mathrm{HS}$$

Universal Algebra Exercises - Sheet 7

Exercise 7.1. Consider the algebra $C_n = (\{0, 1, \ldots, n-1\}, f)$, where f is the unary function $x \mapsto x + 1 \mod n$. Decide for each $n \in \{2, 3, 4, 5, 6\}$ if C_n is simple, subdirectly irreducible or directly indecomposable.

Exercise 7.2. Consider the algebra $\mathbb{A} = (\{0, 1, 2, 3, 4\}, g)$, where g is the unary function given by the following diagram.

$$\begin{tikzcd} & & & 4 \arrow[dl] & \\ 0 \arrow[r] & 1 \arrow[r] & 2 \arrow[rr] & & 3 \arrow[ul] \end{tikzcd}$$

Draw the congruence lattice of $\mathbb{A}$ and then decide whether $\mathbb{A}$ is subdirectly irreducible and or directly indecomposable.

Exercise 7.3. Consider the to algebras $\mathbb{A} := (\{0, 1\}, +, \mathrm{id})$ and $\mathbb{B} = (\{0, 1\}, +, f)$, where $+$ is addition modulo 2 and f is the unary function given by $x \mapsto x + 1 \mod 2$.

(i). Show that $d : \mathbb{B}^2 \to \mathbb{A}, (x, y) \mapsto x + y$ is a surjective homomorphism.

(ii). Show that the congruence lattice of $\mathbb{B}^2$ is

$$\begin{tikzcd}[arrows=-] & 1 \arrow[dl] \arrow[d] \arrow[dr] & \\ \ker(\pi_1) \arrow[dr] & \ker(d) \arrow[d] & \ker(\pi_2) \arrow[dl] \\ & 0 & \end{tikzcd}$$

(iii). Show that $\mathbb{B}^2 \cong \mathbb{B} \times \mathbb{A} \ncong \mathbb{A}^2$ and conclude that direct decompositions are in general not unique.

Exercise 7.4. Find algebras $\mathbb{A}$ and $\mathbb{B}$ such that there are no homomorphisms

$$\mathbb{A} \to \mathbb{A} \times \mathbb{B} \quad \text{and} \quad \mathbb{B} \to \mathbb{A} \times \mathbb{B}$$

Exercise 7.5. Find a proper subdirect composition of the three element lattice into subdirectly irreducible lattices.

Universal Algebra Exercises - Sheet 8

Exercise 8.1. Let p be a prime number. Show that the additive group $\mathbb{Z}$ is a subdirect product of the groups $\mathbb{Z}_{p^k}$.

Exercise 8.2. Let $\mathbb{Q}$ be the additive group of rational numbers and let p be a prime number.

(i). Let $\mathbb{Q}_p := \{a/p^k \mid a \in \mathbb{Z}, k \geq 0\}$. Show that $\mathbb{Q}_p$ is a subgroup of $\mathbb{Q}$ and that $\mathbb{Z}$ is a subgroup of $\mathbb{Q}_p$.

(ii). Let $\mathbb{Z}_{p^\infty} = \mathbb{Q}_p/\mathbb{Z}$. Prove that every element of $\mathbb{Z}_{p^\infty}$ has finite order.

(iii). Let H be a subgroup of $\mathbb{Z}_{p^\infty}$ such that the order of elements in H if bounded. Show that H is a cyclic group of order p^k.

(iv). Show that $\mathbb{Z}_{p^\infty}$ is subdirectly irreducible by showing that the lattice of subgroups is a chain.

$$0 < H_1 < H_2 < \cdots < \mathbb{Z}_{p^\infty}$$

(v). Show that $\mathbb{Z}_{p^\infty}/H_k = \mathbb{Z}_{p^\infty}$ for all k.

Definition. A variety $\mathcal{V}$ is called *finitely generated* if it contains some finite algebras $A_1, \ldots, A_n$ such that $\mathcal{V} = \mathrm{HSP}(A_1, \ldots, A_n)$.

Definition. A variety $\mathcal{V}$ is called *locally finite* if every finitely generated algebra in $\mathcal{V}$ is finite.

Exercise 8.3. Show that every finitely generated variety is locally finite.

Hint: let $B \in \mathrm{HSP}(\mathrm{A}_1, \ldots, \mathrm{A}_\mathrm{n})$ be finitely generated...

Universal Algebra Exercises - Sheet 9

If $\mathcal{V}$ is a variety and X a set, let $F_{\mathcal{V}}(X)$ the free algebra in $\mathcal{V}$ on the set X.

Exercise 9.1. Let $\mathcal{V}$ be the variety of all algebras (A, f) where f is unary and $f^6 = f^2$. Determine and draw $F_{\mathcal{V}}(\{x\})$ and $F_{\mathcal{V}}(\{x, y\})$

Exercise 9.2. Let $\mathcal{S}$ be the variety of semigroups. Show that

$$F_{\mathcal{S}}(X) = (\{\text{nonempty words over } X\}, \circ)$$

where $\circ$ is concatenation of words (for example $(xyz) \circ (zy) = (xyzzy)$).

Exercise 9.3. Let $\mathcal{R}$ be the variety of semigroups satisfying

$$(x \cdot y) \cdot z \approx x \cdot z \quad \text{and} \quad x \cdot x \approx x$$

(i). Describe $F_{\mathcal{R}}(X)$ for any set X.

(ii). Find a natural homomorphism $F_{\mathcal{S}}(X) \to F_{\mathcal{R}}(X)$.

(iii). Generalize (ii) to free algebras of any varieties $\mathcal{W} \subseteq \mathcal{V}$.

Exercise 9.4. Let $\mathcal{V}$ be the variety of distributive lattices.

(i). Describe $F_{\mathcal{V}}(\{x\})$ and $F_{\mathcal{V}}(\{x, y\})$.

(ii). Find an upper bound on the size of $F_{\mathcal{V}}(\{x_1, \ldots, x_n\})$

Remark. The question of whether $F_{\mathcal{V}}(X)$ is always finite for finite X is in general undecidable. It is even unknown if $F_{\mathcal{V}}(\{x, y\})$ is finite if $\mathcal{V}$ is the variety of groups with $(x^5 \approx 1)$. This is part of "Burnsides Problem".

Exercise 9.5. Let $\mathcal{V}$ be the variety of semilattices. Show that

$$F_{\mathcal{V}}(X) = (2^X \setminus \{\emptyset\}, \cup)$$

Universal Algebra Exercises - Sheet 10

Exercise 10.1. Given two varieties of groups $\mathcal{V}$ and $\mathcal{W}$, define

$$\mathcal{V} \cdot \mathcal{W} := \Big\{ G \text{ group} \mid \exists N \trianglelefteq G,\ N \in \mathcal{V},\ G/N \in \mathcal{W} \Big\}.$$

Show that this is a variety of groups.

Exercise 10.2. Let G be a group and let $\mathcal{A}$ be the variety of abelian groups. Let $\lambda^G_{\mathcal{A}}$ be the smallest congruence of G with an abelian quotient.

- Show that the congruence class of the identity element is the subgroup of G generated by elements of the form $[x, y] := xyx^{-1}y^{-1}$.

$$1/\lambda^G_{\mathcal{A}} = \mathrm{Sg}_G([x, y] \mid x, y \in G)$$

- Show that the variety $\mathcal{A} \cdot \mathcal{A}$ is axiomatized by the group laws and $[[x, y], [z, w]] = 1$.

Exercise 10.3. Let $\mathcal{A}_n$ be the variety of abelian groups satisfying $x^n \approx 1$. Show that

- $\mathcal{A}_3 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{ group axioms } \cup \{x^6 \approx 1, \quad [x^2, y^2] \approx 1, \quad [x, y]^3 \approx 1\}\Big)$
- $\mathcal{A}_2 \cdot \mathcal{A}_2 = \mathrm{Mod}\Big(\text{ group axioms } \cup \{(x^2y^2)^2 \approx 1\}\Big)$

Exercise 10.4. Let cRing_n be the vareity of commutative rings satisfying $x^n \approx x$, and let $\mathbb{F}_9$ be the field of order 9. Show that $\mathrm{HSP}(\mathbb{F}_9)$ is axiomatized by the axioms of cRing_9 together with

$$x + x + x \approx 0.$$

Universal Algebra Exercises - Sheet 11

Exercise 11.1. Let $\mathbb{A} = (A, *)$ be an algebra where $A = \{0, 1, 2, 3\}$ and $*$ is defined by the following multiplication table.

$*$	0	1	2	3
0	1	2	1	0
1	0	3	2	3
2	1	0	1	0
3	2	3	2	1

Show that there is no function $f \in \mathrm{Clo}(\mathbb{A})$ satisfying the following.

(i). $f(3, 1, 3, 3, 3) = 0$

(ii). $f(1, 0, 2, 3, 2) = 0$ and $f(1, 0, 0, 3, 2) = 1$

Exercise 11.2. Consider a function $f : \{0, 1\}^n \to \{0, 1\}$ and recall the definition of its dual $f^d(x_1, \ldots x_n) = \neg f(\neg x_1, \ldots, \neg x_n)$.

(i). Show that $\mathrm{Pol}(\neq) = \{f \mid f^d = f\}$.

(ii). For any clone $\mathcal{C}$ on the set $\{0, 1\}$, show that $\mathcal{C}^d := \{f^d \mid f \in \mathcal{C}\}$ is also a clone.

(iii). Find a nice relational description of $\mathcal{C}^d$.

Exercise 11.3. For all numbers $n \geq 1$, define $\mathrm{OR}_n := \{0, 1\}^n \setminus \{(0, \ldots, 0)\}$.

(i). Show that $\mathrm{Clo}(\{0, 1\}; \to) = \mathrm{Pol}(\mathrm{OR_n} \mid \mathrm{n} \geq 1)$

(ii). Show that $\mathrm{Clo}(\{0, 1\}; \to) \subsetneq \mathrm{Pol}(\mathrm{OR}_1, \ldots, \mathrm{OR}_n)$ for any $n \geq 1$.

(iii). Show that $\mathrm{Clo}(\{0, 1\}; \to) \neq \mathrm{Pol}(R_1, \ldots, R_n)$ for any finite set of relations $R_1, \ldots, R_n$.

Universal Algebra Exercises - Sheet 12

Definition. An algebra $\mathbb{A}$ is called *congruence distributive* if its congruence lattice $\mathrm{Con}(\mathbb{A})$ is distributive. A variety $\mathcal{V}$ is called congruence distributive if it contains only congruence distributive algebras.

Theorem (Jónsson, 1967)**.** Let $\mathcal{V}$ be a variety and let $F = F_{\mathcal{V}}(\{x, y, z\})$ be the free algebra with three generators. The following are equivalent.

(i). $\mathcal{V}$ is congruence distributive

(ii). F is congruence distributive

(iii). there is an odd number n and ternary terms $J_0, \dots J_n$ that satisfy the following identities in $\mathcal{V}$:

- $J_0(x, x, y) \approx x$
- $J_n(x, y, y) \approx y$
- $J_i(x, y, x) \approx x$ for all i
- $J_i(x, x, y) \approx J_{i+1}(x, x, y)$ for even i
- $J_i(x, y, y) \approx J_{i+1}(x, y, y)$ for odd i

Exercise 12.1. Show that varieties with a majority term are congruence distributive. Can you do it without Jónssons theorem?

Exercise 12.2. Prove Jónssons theorem.

(i). Note that (i) $\Longrightarrow$ (ii) is trivial.

(ii). Let α, β and γ be the congruences in F generated by a single pair $(x, y), (y, z)$ and (x, z) respectively. Show that $(x, z) \in (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$.

(iii). Conclude that there are elements $j_0, \dots j_n$ in F with

$$x(\alpha \wedge \gamma)j_0(\beta \wedge \gamma) \dots (\alpha \wedge \gamma)j_n(\beta \wedge \gamma)z$$

(iv). Use this to show (ii) $\Longrightarrow$ (iii).

(v). Try to show (iii) $\Longrightarrow$ (i).