Universal Algebra 2 - Exercises 1

Exercise 1.1. Find a digraph that is locally confluent, but not confluent.

Exercise 1.2. Let $\mathcal{E} = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z)\}$ be the theory of semigroups. Show that $D(\mathcal{E})$ is convergent and conclude that the term rewriting algorithm works in this case.

Exercise 1.3. Consider $\mathcal{E} = \{f(f(x)) \approx g(x)\}$.

(i). Show that $D(\mathcal{E})$ is not convergent and try to understand why the term rewriting algorithm doesn't work.

Use the Knuth-Bendix algorithm to find a convergent rewriting system equivalent to $\mathcal{E}$.

(ii). Find a suitable reduction order.

(iii). Find a critical pair of $\mathcal{E}$ and check local confluence.

(iv). What are the normal forms of terms?

Exercise 1.4. Consider $\mathcal{E} = \{(x \cdot y) \cdot z \approx x \cdot (y \cdot z),\ x \cdot x \approx x\}$.

(i). Use the Knuth-Bendix algorithm to expand $\mathcal{E}$ by at least two equations.

(ii). Show that the algorithm enters an infinite loop.

Exercise 1.5. Find a convergent system for the theory of groups. (Hint: think of a normal form and don't use Knuth-Bendix.)

Exercise 1.6. Show that $D(\mathcal{E})$ is not convergent for any axiomatization $\mathcal{E}$ of the variety of commutative rings (Hint: $x + y \approx y + x$). Think of ways to modify the definitions of convergence, normal forms and reductions and describe the resulting term rewriting algorithm.

Universal Algebra 2 - Exercises 2

Exercise 2.1. Show that an abelian algebra $\mathbb{A}$ satisfies the term condition

$$t(x,\bar{u}) \approx t(x,\bar{v}) \implies t(y,\bar{u}) \approx t(y,\bar{v}) \tag{2.1}$$

not only for term operations t, but also for all polynomials $p \in \mathrm{Pol}(\mathbb{A})$. Also, show that it is not enough to satisfy (2.1) only in the case where t is a basic operation of $\mathbb{A}$.

Exercise 2.2. Show that a finite monoid $(M, \cdot, 1)$ is abelian if and only if the multiplication $\cdot$ is a commutative group operation. What if M is infinite?

Exercise 2.3. Let A be a 4-element set, fix $0 \in A$ and let $(A, +_1) \cong \mathbb{Z}_4$ and $(A, +_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ be the two abelian group operations on A with neutral element 0. Show that $(A, +_1, +_2)$ is not an abelian algebra.

Exercise 2.4. Let $(R, +, 0, -, \cdot)$ be a commutative ring. Recall that congruences α are one-to-one with ideals I, using $I_\alpha = [0]_\alpha$. Show that α centralizes β if and only if $I_\alpha \cdot I_\beta = 0$. More generally, show that $I_\alpha \cdot I_\beta = I_{[\alpha,\beta]}$.

Exercise 2.5. Show the following properties of the centralizer relation C:

- $C(\alpha, \beta; \alpha)$ and $C(\alpha, \beta; \beta)$
- Let Γ be a set of congruences. If $C(\alpha, \beta; \gamma)$ for all $\gamma \in \Gamma$, then $C(\alpha, \beta; \bigwedge \Gamma)$.

Universal Algebra 2 - Exercises 3

Exercise 3.1. Consider the Loop $(L, \cdot)$ with universe $\mathbb{Z}_4 \times \mathbb{Z}_2$ given by the multiplication

$$(a, b) \cdot (c, d) = (a + c, b + d) \quad \text{unless } b = d = 1$$
$$(a, 1) \cdot (c, 1) = (a * c, 0) \quad \text{where}$$

$*$	0	1	2	3
0	1	0	2	3
1	0	2	3	1
2	2	3	1	0
3	3	1	0	2

Consider the map $f : L \to \mathbb{Z}_2, (a, b) \mapsto b$, its kernel α and the α-block N of $(0, 0)$.

- Show that f is a homomorphism and that N is an abelian subloop.
- Show that α is not an abelian congruence, i.e. $C(\alpha, \alpha, 0)$ does not hold.

Exercise 3.2. Prove that the polynomial equivalence problem of nilpotent rings is solvable in polynomial time. Hint: Look at Example 2.26 in the script.

Exercise 3.3. We call an algebra k-supernilpotent if every $k+1$-ary absorbing polynomial is constant. Consider the algebra $(\mathbb{Z}_9, +, 0, -, f_n(x_1, \dots, x_n) \mid n \in \mathbb{N})$ where $f_n(x_1, \dots, x_n) = 3 \cdot x_1 \cdot \dots \cdot x_n$. Show that this algebra is 2-nilpotent but no k supernilpotent for any k

Universal Algebra 2 - Exercises 4

Exercise 4.1. Recall that a variety is already finitely based, if it has definable principal congruences and finitely many subdirectly irreducibles (up to isomorphism). Consider commutative rings R that satisfy equation $x^n \approx x$.

- Show that every subdirecly irreducible such ring is a field of order d, where $d - 1 \mid n - 1$.
- Conclude that $\mathrm{HSP}(R)$ is finitely based.

Exercise 4.2. Let $\mathbb{A}$ and $\mathbb{B}$ be two algebras of finite type on the same domain with $\mathrm{Clo}(\mathbb{A}) = \mathrm{Clo}(\mathbb{B})$. Show that $\mathbb{A}$ is finitely based if and only if $\mathbb{B}$ is finitely based. Is this still true if we do not assume finite type?

Universal Algebra 2 - Exercises 5

Exercise 5.1. Let $\mathbb{A}$ be a relational structure in signature τ. Show that the following decision problem is equivalent to CSP($\mathbb{A}$):

- INPUT: a τ structure $\mathbb{X}$
- QUESTION: is there a homomorphism $\mathbb{X} \to \mathbb{A}$?

Conclude that if there are homomorphisms $\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$, then the CSPs of $\mathbb{A}$ and $\mathbb{B}$ are the same.

Exercise 5.2. Consider the computational problem nCOLOR, of coloring a given graph with n many colors.

- Find a relational structure $\mathbb{A}$ such that nCOLOR = CSP($\mathbb{A}$).
- Find a polynomial time reduction of 3COLOR to nCOLOR and conclude that nCOLOR is NP-hard.

Exercise 5.3. Let A be a finite set. Show that a function $f : A^n \to A$ preserves all relations on A if and only if it is a projection.

Exercise 5.4. Recall the structure $\mathbb{A} = (\{0,1\}; R_{000}, R_{001}, R_{011}, R_{111})$ and that CSP($\mathbb{A}$) = 3SAT. Show that all polymorphisms of $\mathbb{A}$ are projections. (Hint: what can you *pp*-define from the relations in $\mathbb{A}$?)

Universal Algebra 2 - Exercises 6

Exercise 6.1. Let $\mathbb{A}$ and $\mathbb{B}$ be two homomorphically equivalent relational structures ($\mathbb{A} \to \mathbb{B}$ and $\mathbb{B} \to \mathbb{A}$). Show that there is a minion homomorphism $\mathrm{Pol}(\mathbb{A}) \xrightarrow{\text{minion}} \mathrm{Pol}(\mathbb{B})$.

Exercise 6.2. Let $\mathbb{A} = (\{0\}, =)$ and $\mathbb{B} = (\{0,1\}, \leq)$. Show that $\mathbb{A}$ and $\mathbb{B}$ are homomorphically equivalent but that there is no clone homomorphism $\mathrm{Pol}(\mathbb{A}) \xrightarrow{\text{clone}} \mathrm{Pol}(\mathbb{B})$. (Hint: show that $\mathrm{Pol}(\mathbb{B})$ does not contain a Maltsev term)

Exercise 6.3. Let $R \subseteq A^n$ be a relation compatible with a majority polymorphisms $m : A^3 \to A$.

$$m(x,x,y) \approx m(x,y,x) \approx m(y,x,x) \approx x$$

Denote by $\pi_{i,j}(R)$ the projections of R to the coordinates i, i ($1 \leq i, j \leq n$).

$$\pi_{i,j} = \{(a_i, a_j) \mid (a_1, \ldots, a_n) \in R\}$$

Show that R is determined by these binary projections, i.e.

$$(a_1, \ldots, a_n) \in R \iff \forall i, j \quad (a_i, a_j) \in \pi_{i,j}(R)$$

Conclude that R is *pp*-definable from binary relations.

Exercise 6.4. Find a finite set of relations $\{R_1, \ldots, R_n\}$ on the set $\{0,1\}$ such that $\mathrm{Pol}(R_1, \ldots, R_n)$ is the clone generated by the unique majority operation on $\{0,1\}$.

Universal Algebra 2 - Exercises 7

Exercise 7.1. Show that all idempotent polymorphisms of $K_3 = (\{0,1,2\}, \neq)$ are projections using the following steps:

(i). Show that every binary polymorphism is a projection.

For an n-ary polymorphism f and $1 \leq i \leq n$, define the minors $f_i(x,y) = f(x,\dots,x,y,x,\dots,x)$, where the only y is in the i-th position. Observe that every f_i is a projection.

(ii). Show that $f_i(x,y) \approx y$ holds for at most one i.

(iii). Show that $f_i(x,y) \approx x$ cannot hold for all i. (Hint: try to *pp*-define the relation $N = \{0,1,3\}^3 \setminus \{(0,0,0),(1,1,1),(2,2,2)\}$)

(iv). Show that $f_i(x,y) = y$ implies that f is the i-th projection.